UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education
Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1

May/June 2005

3 hours

Additional materials: Answer Booklet/Paper
Graph paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

This document consists of **5** printed pages and **3** blank pages.

1 Use the method of differences to find S_N, where

$$S_N = \sum_{n=N}^{N^2} \frac{1}{n(n+1)}.$$ [3]

Deduce the value of $\lim_{N \to \infty} S_N$. [1]

2 Given that

$$x = t - \sin t \qquad \text{and} \qquad y = 1 - \cos t,$$

where $0 < t < 2\pi$, show that

$$\frac{d^2y}{dx^2} = -\tfrac{1}{4}\operatorname{cosec}^4\left(\tfrac{1}{2}t\right).$$ [5]

3 The points A, B, C have position vectors $a\mathbf{i}$, $b\mathbf{j}$, $c\mathbf{k}$ respectively, where a, b, c are all positive. The plane containing A, B, C is denoted by Π.

(i) Find a vector perpendicular to Π. [3]

(ii) Find the perpendicular distance from the origin to Π, in terms of a, b, c. [3]

4 Show that the sum of the cubes of the roots of the equation

$$x^3 + \lambda x + 1 = 0$$

is -3. [3]

Show also that there is no real value of λ for which the sum of the fourth powers of the roots is negative. [3]

5 A curve is defined parametrically by

$$x = t - 8t^{\frac{1}{2}} \qquad \text{and} \qquad y = \tfrac{16}{3}t^{\frac{3}{4}}.$$

The arc of this curve joining the point where $t = 1$ to the point where $t = 4$ is denoted by C.

(i) Show that the length of C is 11. [5]

(ii) Find, correct to 3 significant figures, the area of the surface generated when C is rotated through one complete revolution about the x-axis. [3]

6 The curve C has polar equation

$$r = \frac{\pi - \theta}{\theta},$$

where $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$.

(i) Draw a sketch of C. [3]

(ii) Show that the area of the region bounded by the line $\theta = \frac{1}{2}\pi$ and C is

$$\pi\left(\tfrac{3}{4} - \ln 2\right).$$ [5]

7 The planes Π_1 and Π_2 have equations

$$x + 2y - 3z + 4 = 0 \qquad \text{and} \qquad 2x + y - 4z - 3 = 0$$

respectively. Show that, for all values of λ, every point which is in both Π_1 and Π_2 is also in the plane

$$x + 2y - 3z + 4 + \lambda(2x + y - 4z - 3) = 0.$$ [2]

The planes Π_1 and Π_2 meet in the line l.

(i) Find the equation of the plane Π_3 which passes through l and the point whose position vector is $a\mathbf{k}$. [3]

(ii) Find the value of a if Π_2 is perpendicular to Π_3. [3]

8 The integral I_n, where n is a non-negative integer, is defined by

$$I_n = \int_0^1 e^{-x}(1 - x)^n \, dx.$$

(i) Show that $I_{n+1} = 1 - (n + 1)I_n$. [3]

(ii) Use induction to show that I_n is of the form $A_n + B_n e^{-1}$, where A_n and B_n are integers. [4]

(iii) Express B_n in terms of n. [2]

9 Find the eigenvalues and a corresponding set of eigenvectors of the matrix **M** given by

$$\mathbf{M} = \begin{pmatrix} a & 2 & 1 \\ 0 & b & -1 \\ 0 & 0 & c \end{pmatrix},$$

where a, b, c are all different. [6]

Find a matrix **P** and a diagonal matrix **D** such that

$$(\mathbf{M} - k\mathbf{I})^n = \mathbf{PDP}^{-1},$$

where **I** is the identity matrix, k is a constant scalar and n is a positive integer. [4]

[You are not required to evaluate $\mathbf{P}^{-1}$.]

10 **(i)** Write down, in any form, all the complex roots of the equation

$$w^{12} = 1.$$ [2]

(ii) Explain why the equation

$$(z + 2)^{12} = z^{12} \qquad (*)$$

has exactly 10 non-real roots and show that they may be expressed in the form

$$-1 - \mathrm{i}\cot\left(\tfrac{1}{12}k\pi\right),$$

where $k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. [6]

(iii) Show that

$$\left\{-1 - \mathrm{i}\cot\left(\tfrac{1}{12}k\pi\right)\right\}\left\{-1 + \mathrm{i}\cot\left(\tfrac{1}{12}k\pi\right)\right\} = \operatorname{cosec}^2\left(\tfrac{1}{12}k\pi\right).$$ [1]

(iv) Given that the product of the roots of (*) is $-\frac{512}{3}$, find the value of

$$\sin^2\left(\tfrac{1}{12}\pi\right)\sin^2\left(\tfrac{2}{12}\pi\right)\sin^2\left(\tfrac{3}{12}\pi\right)\sin^2\left(\tfrac{4}{12}\pi\right)\sin^2\left(\tfrac{5}{12}\pi\right).$$ [2]

11 The matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & -1 \\ 2 & 2 & \theta \end{pmatrix}.$$

Find the rank of $\mathbf{A}$, distinguishing between the cases $\theta \neq 1$ and $\theta = 1$. [4]

Consider the system S of equations:

$$\begin{aligned} x + 3y + 2z &= 1, \\ x - \ \ y - \ \ z &= 0, \\ 2x + 2y + \theta z &= 3\theta + \phi - 2. \end{aligned}$$

(i) Show that if $\theta \neq 1$ then S has a unique solution. Find this solution in the case $\phi = 0$. [3]

(ii) Show that if $\theta = 1$ and $\phi = 0$ then S has an infinite number of solutions. [3]

(iii) Show that if $\theta = 1$ and $\phi \neq 0$ then S has no solution. [2]

12 Answer only **one** of the following two alternatives.

EITHER

The curve Γ, which has equation

$$y = \frac{ax^2 + bx + c}{x^2 + px + q},$$

has asymptotes $x = 1$, $x = 4$ and $y = 2$. Find the values of a, p and q. [4]

It is given that Γ has a stationary point at $x = 2$.

(i) Find the value of c. [3]

(ii) Show that if $b \neq -10$ then Γ has exactly 2 stationary points. [2]

(iii) Draw a sketch of Γ for the case where $b = -6$. [4]

OR

It is given that

$$\frac{d^2y}{dx^2} + (2a - 1)\frac{dy}{dx} + a(a - 1)y = 2a - 1 + a(a - 1)x,$$

where a is a constant. Find y in terms of a and x, given that y and $\frac{dy}{dx}$ are both zero when $x = 0$. [8]

Hence show that if $a > 1$ then $y \approx x$ as $x \to \infty$. [2]

It is given that

$$\frac{d^2z}{dx^2} + (2a - 1)\frac{dz}{dx} + a(a - 1)z = e^x,$$

where the constant a is positive. Find $\lim_{x \to \infty} e^{-x}z$. [3]

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