**CAMBRIDGE INTERNATIONAL EXAMINATIONS**

**General Certificate of Education Advanced Level**

# FURTHER MATHEMATICS 9231/1

PAPER 1

**OCTOBER/NOVEMBER SESSION 2002**

3 hours

Additional materials:
- Answer paper
- Graph paper
- List of Formulae (MF10)

**TIME** 3 hours

## INSTRUCTIONS TO CANDIDATES

Write your name, Centre number and candidate number in the spaces provided on the answer paper/answer booklet.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

## INFORMATION FOR CANDIDATES

The number of marks is given in brackets [ ] at the end of each question or part question.

The use of a calculator is expected, where appropriate.

Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.

You are reminded of the need for clear presentation in your answers.

**This question paper consists of 5 printed pages and 3 blank pages.**



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1 Given that

$$u_n = e^{nx} - e^{(n+1)x},$$

find $\sum_{n=1}^{N} u_n$ in terms of $N$ and $x$. [2]

Hence determine the set of values of $x$ for which the infinite series

$$u_1 + u_2 + u_3 + \ldots$$

is convergent and give the sum to infinity for cases where this exists. [3]

2 The equation

$$x^4 + x^3 + Ax^2 + 4x - 2 = 0,$$

where $A$ is a constant, has roots $\alpha, \beta, \gamma, \delta$. Find a polynomial equation whose roots are

$$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}.$$ [2]

Given that

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2},$$

find the value of $A$. [3]

3 It is given that, for $n = 0, 1, 2, 3, \ldots,$

$$a_n = 17^{2n} + 3(9)^n + 20.$$

Simplify $a_{n+1} - a_n$, and hence prove by induction that $a_n$ is divisible by 24 for all $n \geqslant 0$. [6]

4 It is given that, for $n \geqslant 0$,

$$I_n = \int_0^1 x^n e^{-x^2}\, dx.$$

**(i)** Find $I_1$ in terms of e. [1]

**(ii)** Show that

$$I_{n+2} = \frac{n+1}{2} I_n - \frac{1}{2e}.$$ [3]

**(iii)** Find $I_5$ in terms of e. [3]

**5** The curve $C$ has polar equation $r\theta = 1$, for $0 < \theta \leqslant 2\pi$.

**(i)** Use the fact that $\dfrac{\sin\theta}{\theta}$ tends to 1 as $\theta$ tends to 0 to show that the line with cartesian equation $y = 1$ is an asymptote to $C$. [2]

**(ii)** Sketch $C$. [1]

The points $P$ and $Q$ on $C$ correspond to $\theta = \frac{1}{6}\pi$ and $\theta = \frac{1}{3}\pi$ respectively.

**(iii)** Find the area of the sector $OPQ$, where $O$ is the origin. [3]

**(iv)** Show that the length of the arc $PQ$ is

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\sqrt{(1+\theta^2)}}{\theta^2}\,\mathrm{d}\theta.$$ [2]

**6** A curve has equation $x^3 + xy^2 - y^3 = 3$.

**(i)** Show that there is no point of the curve at which $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$. [4]

**(ii)** Find the values of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point $(1, -1)$. [5]

**7** Given that $z = \cos\theta + \mathrm{i}\sin\theta$, show that

**(i)** $z - \dfrac{1}{z} = 2\mathrm{i}\sin\theta$, [1]

**(ii)** $z^n + z^{-n} = 2\cos n\theta$. [2]

Hence show that

$$\sin^6\theta = \tfrac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta).$$ [3]

Find a similar expression for $\cos^6\theta$, and hence express $\cos^6\theta - \sin^6\theta$ in the form $a\cos 2\theta + b\cos 6\theta$. [3]

**8** The value of the assets of a large commercial organisation at time $t$, measured in years, is \$$(10^8y + 10^9)$. The variables $y$ and $t$ are related by the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 5\frac{\mathrm{d}y}{\mathrm{d}t} + 6y = 15\cos 3t - 3\sin 3t.$$

Find $y$ in terms of $t$, given that $y = 3$ and $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -2$ when $t = 0$. [9]

Show that, for large values of $t$, the value of the assets is less than \$$9.5 \times 10^8$ for about a third of the time. [3]

**9** The planes $\Pi_1$ and $\Pi_2$, which meet in the line $l$, have vector equations

$$\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \theta_1(2\mathbf{i} + 3\mathbf{k}) + \phi_1(-4\mathbf{j} + 5\mathbf{k}),$$
$$\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \theta_2(3\mathbf{j} + \mathbf{k}) + \phi_2(-\mathbf{i} + \mathbf{j} + 2\mathbf{k}),$$

respectively. Find a vector equation of the line $l$ in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. [5]

Find a vector equation of the plane $\Pi_3$ which contains $l$ and which passes through the point with position vector $4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. Find also the equation of $\Pi_3$ in the form $ax + by + cz = d$. [4]

Deduce, or prove otherwise, that the system of equations

$$\begin{aligned} 6x - 5y - 4z &= -32, \\ 5x - y + 3z &= 24, \\ 9x - 2y + 5z &= 40, \end{aligned}$$

has an infinite number of solutions. [3]

**10** The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{H}$, where

$$\mathbf{H} = \begin{pmatrix} 1 & 2 & -3 & -5 \\ -1 & 4 & 5 & 1 \\ 2 & 3 & 0 & -3 \\ -3 & 5 & 7 & 2 \end{pmatrix}.$$

**(i)** Find the dimension of the range space of T. [3]

**(ii)** Find a basis for the null space of T. [3]

**(iii)** It is given that $\mathbf{x}$ satisfies the equation

$$\mathbf{Hx} = \begin{pmatrix} 2 \\ -10 \\ -1 \\ -15 \end{pmatrix}.$$

Using the fact that

$$\mathbf{H}\begin{pmatrix} 1 \\ -3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \\ -1 \\ -15 \end{pmatrix},$$

find the least possible value of $|\mathbf{x}|$. [7]

[For the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $|\mathbf{x}| = \sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2)}$.]

**11** Answer only **one** of the following two alternatives.

**EITHER**

The vector **e** is an eigenvector of the square matrix **G**. Show that

**(i)** **e** is an eigenvector of $\mathbf{G} + k\mathbf{I}$, where $k$ is a scalar and **I** is an identity matrix,

**(ii)** **e** is an eigenvector of $\mathbf{G}^2$.

[5]

Find the eigenvalues, and corresponding eigenvectors, of the matrices **A** and $\mathbf{B}^2$, where

$$\mathbf{A} = \begin{pmatrix} 3 & -3 & 0 \\ 1 & 0 & 1 \\ -1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -5 & -3 & 0 \\ 1 & -8 & 1 \\ -1 & 3 & -6 \end{pmatrix}.$$ [9]

**OR**

The curve $C$ has equation

$$y = \frac{(x-a)(x-b)}{x-c},$$

where $a$, $b$, $c$ are constants, and it is given that $0 < a < b < c$.

**(i)** Express $y$ in the form

$$x + P + \frac{Q}{x-c},$$

giving the constants $P$ and $Q$ in terms of $a$, $b$ and $c$. [3]

**(ii)** Find the equations of the asymptotes of $C$. [2]

**(iii)** Show that $C$ has two stationary points. [5]

**(iv)** Given also that $a + b > c$, sketch $C$, showing the asymptotes and the coordinates of the points of intersection of $C$ with the axes. [4]

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