BOARD OF STUDIES
NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1998
MATHEMATICS
4 UNIT (ADDITIONAL)

Time allowed—Three hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a SEPARATE Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1. Use a SEPARATE Writing Booklet. **Marks**

(a) Evaluate $\int_0^3 \frac{6}{9+x^2}\,dx$. **2**

(b) Find $\int x^2 \ln x \; dx$. **2**

(c) Find $\int \frac{\sin^3 x}{\cos^2 x}\,dx$. **3**

(d) Using the substitution $u^2 = 4 - x^2$, or otherwise, evaluate $\int_0^2 x^3\sqrt{4-x^2}\,dx$. **4**

(e) (i) Find the remainder when $x^2 + 6$ is divided by $x^2 + x - 6$. **4**

(ii) Hence, find $\int \frac{x^2+6}{x^2+x-6}\,dx$.

QUESTION 2. Use a SEPARATE Writing Booklet. **Marks**

(a) Evaluate i^{1998}. **1**

(b) Let $z = \dfrac{18+4i}{3-i}$. **5**

(i) Simplify $(18+4i)\overline{(3-i)}$.

(ii) Express z in the form $a + ib$, where a and b are real numbers.

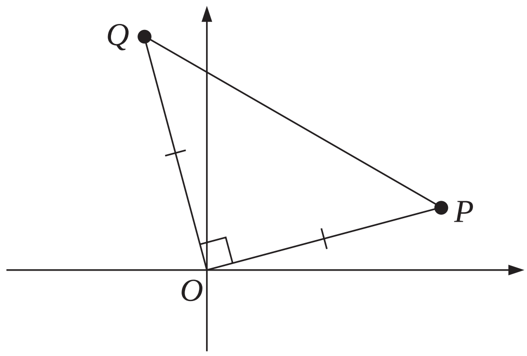
(iii) Hence, or otherwise, find $|z|$ and $\arg(z)$.

(c) Sketch the region in the complex plane where the inequalities **2**

$$|z-2+i| \le 2 \quad \text{and} \quad \text{Im}(z) \ge 0$$

both hold.

(d) **1**



The points P and Q in the complex plane correspond to the complex numbers z and w respectively. The triangle OPQ is isosceles and $\angle POQ$ is a right angle.

Show that $z^2 + w^2 = 0$.

(e) (i) By solving the equation $z^3 + 1 = 0$, find the three cube roots of -1. **6**

(ii) Let λ be a cube root of -1, where λ is not real. Show that $\lambda^2 = \lambda - 1$.

(iii) Hence simplify $(1 - \lambda)^6$.

QUESTION 3. Use a SEPARATE Writing Booklet. **Marks**

(a) Let $f(x) = x - \dfrac{4}{x}$. Provide separate half-page sketches of the graphs of the following functions. **6**

(i) $y = f(x)$

(ii) $y = \sqrt{f(x)}$

(iii) $y = e^{f(x)}$

Label each graph with its equation.

(b) Let $I_n = \displaystyle\int_1^e (\ln x)^n \, dx$. **4**

(i) Show that $I_n = e - nI_{n-1}$ for $n = 1, 2, 3, \ldots$

(ii) Hence evaluate I_4.

(c) The population P of a town decreases at a rate proportional to the number by which the population exceeds 1000. Thus **5**

$$\frac{dP}{dt} = -k(P - 1000) .$$

(i) Show that $P = 1000 + Ae^{-kt}$, where A and k are constants, is a solution of this equation.

(ii) Initially the population of the town was 2500. Ten years later, it had fallen to 1900.

When will the population be 1500?

(iii) What does this mathematical model predict about the population of the town in the long term?

QUESTION 4. Use a SEPARATE Writing Booklet. **Marks**

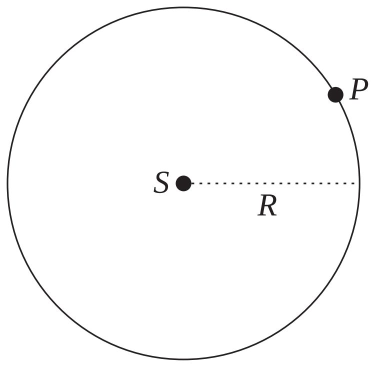
(a) (i) Suppose that k is a double root of the polynomial equation $f(x) = 0$. Show that $f'(k) = 0$. **7**

(ii) What feature does the graph of a polynomial have at a root of multiplicity 2?

(iii) The polynomial $P(x) = ax^7 + bx^6 + 1$ is divisible by $(x-1)^2$. Find the coefficients a and b.

(iv) Let $E(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$. Prove $E(x) = 0$ has no double roots.

(b) **5**



A planet P of mass m kilograms moves in a circular orbit of radius R metres around a star S. Coordinate axes are taken in the plane of the motion, centred at S. The position of the planet at time t seconds is given by the equations

$$x = R\cos\frac{2\pi t}{T} \quad \text{and} \quad y = R\sin\frac{2\pi t}{T},$$

where T is a constant.

(i) Show that the planet is subject to a force of constant magnitude, F newtons.

(ii) It is known that the magnitude of the gravitational force pulling the planet towards the star is given by

$$F = \frac{GMm}{R^2},$$

where G is a constant and M is the mass of the star S in kilograms. Find an expression for T in terms of R, M and G.

Question 4 continues on page 6

QUESTION 4. (Continued) **Marks**

(c) An urn contains 3 red balls and w white balls. **3**

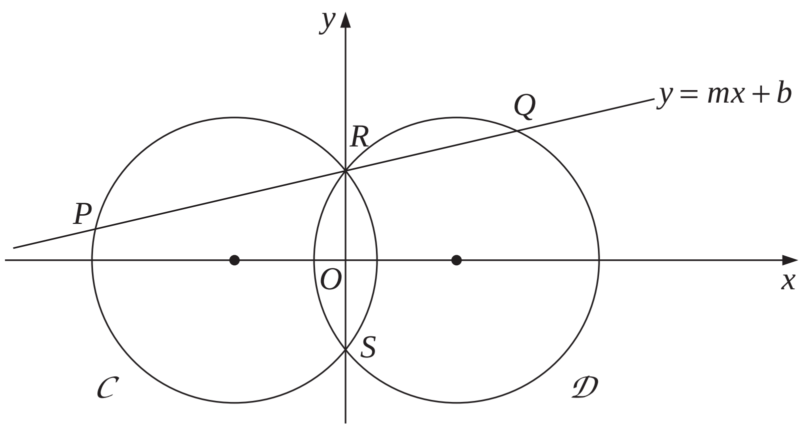
Sue draws two balls together from the urn. The probability that they have the same colour is $\frac{1}{2}$.

Bill adopts a different procedure. He draws one ball from the urn, notes its colour and replaces it. He then draws a second ball from the urn and notes its colour. The probability that both balls have the same colour is now $\frac{5}{8}$.

Find all possible values of w.

QUESTION 5. Use a SEPARATE Writing Booklet.

(a) **6**



The diagram shows the circles $\mathcal{C}$: $(x+a)^2 + y^2 = a^2 + b^2$ and $\mathcal{D}$: $(x-a)^2 + y^2 = a^2 + b^2$, which meet at the points $R(0, b)$ and $S(0, -b)$. The straight line $y = mx + b$ meets the circles at P, Q and R, as shown in the diagram.

(i) Show that the x coordinate of the point P is $\dfrac{-2(a+mb)}{1+m^2}$.

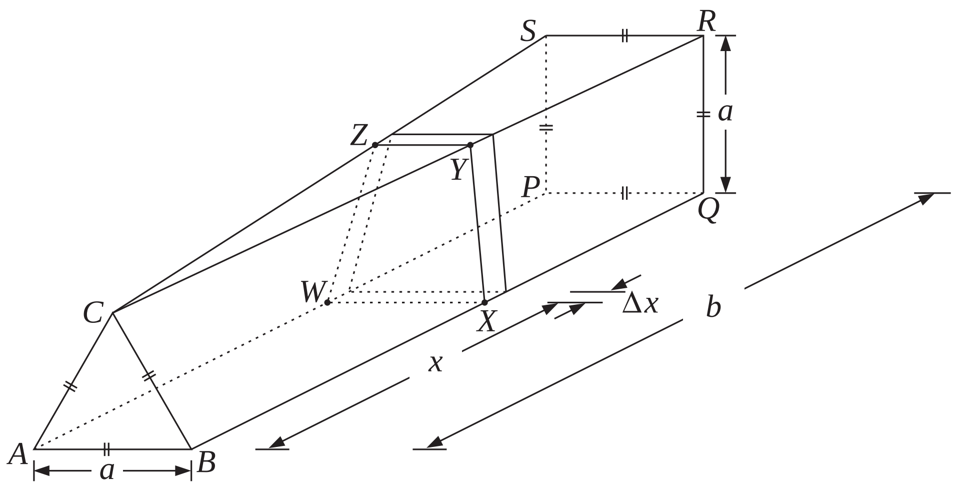
(ii) Find the x coordinate of the point Q.

(iii) Hence find the equation of the locus of the midpoint of PQ as the slope of the straight line through R varies. Describe this locus geometrically.

Question 5 continues on page 7

QUESTION 5. (Continued) **Marks**

(b) **9**



The diagram shows a sandstone solid with rectangular base $ABQP$ of length b metres and width a metres. The end $PQRS$ is a square, and the other end ABC is an equilateral triangle. Both ends are perpendicular to the base.

Consider the slice of the solid with face $WXYZ$ and thickness Δx metres, as shown in the diagram. The slice is parallel to the ends and $AW = BX = x$ metres.

(i) Find the height of the equilateral triangle ABC.

(ii) Given that the triangles CRS and CYZ are similar, find YZ in terms of a, b and x.

(iii) Let the perpendicular height of the trapezium $WXYZ$ be h metres. Show that

$$h = \frac{a}{2}\left[\sqrt{3} + \left(2 - \sqrt{3}\right)\frac{x}{b}\right].$$

(iv) Hence show that the cross-sectional area of $WXYZ$ is given by

$$\frac{a^2}{4b^2}\left[\left(2 - \sqrt{3}\right)x + b\sqrt{3}\right](b + x).$$

(v) Find the volume of the solid.

QUESTION 6. Use a SEPARATE Writing Booklet. **Marks**

(a) Consider the following statements about a polynomial $Q(x)$. **2**

(i) If $Q(x)$ is even, then $Q'(x)$ is odd.

(ii) If $Q'(x)$ is even, then $Q(x)$ is odd.

Indicate whether each of these statements is true or false. Give reasons for your answers.

(b) The probability that n accidents occur at a given intersection during a year is **6**

$$P_n = e^{-2\cdot 6}\frac{(2\cdot 6)^n}{n!}, \qquad n = 0, 1, 2, \ldots.$$

(i) Find the probability that no accidents occur at the intersection in a given year. Give your answer correct to three decimal places.

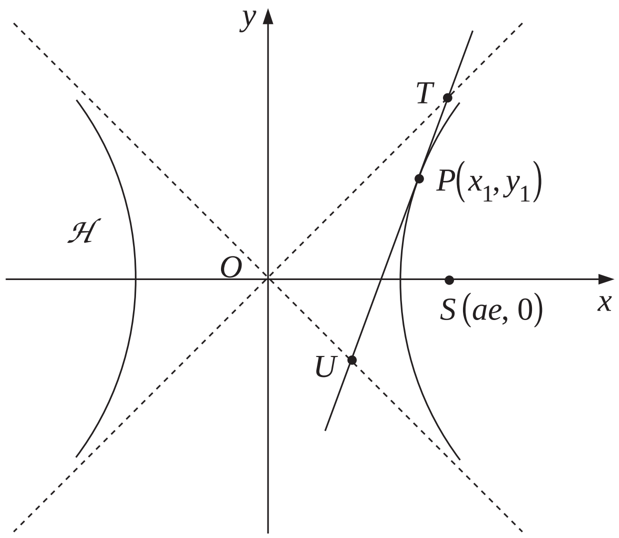
(ii) What is the probability that, in a given ten-year period, there are at least 2 years in which no accidents occur at the intersection? Give your answer correct to three decimal places.

(iii) By considering values of n for which $\frac{P_{n+1}}{P_n} \geq 1$, determine the most likely number of accidents in a given one-year period.

Question 6 continues on page 9

QUESTION 6. (Continued) **Marks**

(c) **7**



The point $S(ae, 0)$ is a focus of the hyperbola $\mathscr{H}$: $x^2 - y^2 = a^2$. The tangent to the hyperbola at a point $P(x_1, y_1)$ meets the asymptotes of $\mathscr{H}$ in T and U, as shown in the diagram.

(i) Show that the equation of the tangent TU is

$$x_1 x - y_1 y = a^2.$$

(ii) Show that the gradient of SU is

$$\frac{a}{e(x_1 + y_1) - a}.$$

(iii) Let $\angle UST = \theta$. Show that $\tan\theta = -1$.

QUESTION 7. Use a SEPARATE Writing Booklet. **Marks**

(a) Let $P(z) = z^8 - \frac{5}{2}z^4 + 1$. The complex number w is a root of $P(z) = 0$. **6**

(i) Show that iw and $\frac{1}{w}$ are also roots of $P(z) = 0$.

(ii) Find one of the roots of $P(z) = 0$ in exact form.

(iii) Hence find all the roots of $P(z) = 0$.

(b) (i) Differentiate $\sin^{-1}(u) - \sqrt{1-u^2}$. **3**

(ii) Hence show that $\int_0^{\alpha} \left(\frac{1+u}{1-u}\right)^{\frac{1}{2}} du = \sin^{-1}\alpha + 1 - \sqrt{1-\alpha^2}$ for $0 < \alpha < 1$.

(c) A bead of mass m slides along a wire in the shape of the curve **6**

$$y = \tfrac{3}{2}x^{\frac{2}{3}}, \qquad \text{where } 0 \le x \le 1.$$

At time t, the bead is at $\left(x(t), y(t)\right)$, and its velocity is $\left(\dot{x}(t), \dot{y}(t)\right)$. The motion of the bead is governed by the equations

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy = E\,,$$

where E and g are constants, and

$$\dot{x} = x^{\frac{1}{3}}\dot{y}\,.$$

When $t = 0$, the bead is released from rest at the point $\left(1, \tfrac{3}{2}\right)$. It accelerates along the wire towards the origin, where it arrives at time t_1.

(i) Find E, and show that $\dot{y}^2 = \dfrac{3g(3-2y)}{3+2y}$.

(ii) Find $\dot{x}(t_1)$ and $\dot{y}(t_1)$.

(iii) Using the result of part (b), or otherwise, find the time it takes for the bead to travel from $\left(\tfrac{1}{8}, \tfrac{3}{8}\right)$ to the origin.

QUESTION 8. Use a SEPARATE Writing Booklet. **Marks**

(a) The numbers p, q and s are fixed and positive. Also $p > 1$, $q > 1$ and $p = \dfrac{q}{q-1}$. **8**

(i) What positive value of t minimises the expression

$$f(t) = \frac{s^p}{p} + \frac{t^q}{q} - st \,?$$

(ii) Show that for all $t > 0$,

$$\frac{s^p}{p} + \frac{t^q}{q} \geq st\,.$$

(iii) Prove by induction that

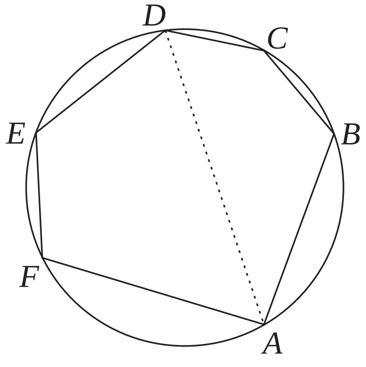
$$\left(x_1 x_2 \cdots x_n\right)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for all $x_1, \ldots, x_n > 0$.

(iv) Deduce that, for all $y_1, y_2, \ldots, y_n > 0$,

$$\frac{y_1}{y_2} + \frac{y_2}{y_3} + \cdots + \frac{y_{n-1}}{y_n} + \frac{y_n}{y_1} \geq n\,.$$

(b) **7**



$ABCDEF$ is a cyclic hexagon.

(i) Show that $\angle DAB + \angle BCD = \angle ABC + \angle CDA$.

(ii) Show that $\angle FAD + \angle DEF = \angle EFA + \angle ADE$.

(iii) Deduce that $\angle ABC - \angle BCD + \angle CDE - \angle DEF + \angle EFA - \angle FAB = 0$.

(iv) State and prove a similar result for a cyclic octagon.

(v) Formulate a similar result for a cyclic n-gon.

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$